SIMPLE Algorithm for Two-Dimensional Flow - Fluid Flow and Heat Transfer

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Two-dimensional, transient, incompressible laminar flow neglecting body force:

(1) Continuity: \[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

(2) Momentum: \[
\begin{align*}
\text{x-direction:} & \quad \frac{\partial (\rho u)}{\partial t} + [ \frac{\partial (\rho u^2)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} ] = - \frac{\partial P_e}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
\text{y-direction:} & \quad \frac{\partial (\rho v)}{\partial t} + [ \frac{\partial (\rho uv)}{\partial x} + \frac{\partial (\rho v^2)}{\partial y} ] = - \frac{\partial P_e}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\end{align*}
\]

(3) Energy: \[
\frac{\partial T}{\partial t} + [ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} ] = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)
\]

For example: the geometry of the original problem (2-D channel) is

\[ u_\infty \]

\[ v \]

\[ L_x \]

\[ L_y \]
• The SIMPLE (semi-implicit method for pressure-linked equations) algorithm can only be used to solve the velocity distribution of **steady flow** (穩態流).

• It can be arranged into a **finite-difference scheme** or a **finite-volume scheme**.

• In this class, a **finite-difference scheme** is developed for the SIMPLE algorithm.

• A large number of iterations involved in the calculation, thus the method is quite **time-consuming**.

• In the method, the Navier-Stokes equations for unsteady flow was used. Time progressing in calculation is treated as iteration steps.
• As every grid point in the flow field is satisfied with the continuity equation and the momentum equations, the acquired velocity distribution is the true velocity distribution.

• The intermittent result during the iterations does not possess any physical meaning.

• The SIMPLE algorithm can also be used to solve the velocity distribution of three-dimensional flow whether the flow is laminar or turbulent.
Finite-Difference Approach (Staggered Grid - 交錯網格)

Note:  
- pressure grid points: magenta circles
- x-dir. velocity (u) grid points: green circles
- y-dir. velocity (v) grid points: blue circles

Grid points:
- (i+1, j+1)
- (i+1/2, j+1)
- (i+1/2, j)
- (i+1/2, j-1)
- (i+1, j-1/2)
- (i+3/2, j)
- (i+1, j+1/2)
- (i, j+1/2)
- (i-1, j)
- (i-1/2, j)
- (i, j)
- (i, j-1/2)
- (i+1, j-1/2)

Grid spacing:
- dx
- dy
At point "a": \( \vec{v}_{i+1/2, j+1/2} \equiv \frac{1}{2}(v_{i,j+1/2} + v_{i+1,j+1/2}) \)

At point "b": \( \vec{v}_{i+1/2, j-1/2} \equiv \frac{1}{2}(v_{i,j-1/2} + v_{i+1,j-1/2}) \)

At point "c": \( \vec{u}_{i-1/2, j+1/2} \equiv \frac{1}{2}(u_{i-1/2,j} + u_{i-1/2,j+1}) \)

At point "d": \( \vec{u}_{i+1/2, j+1} \equiv \frac{1}{2}(u_{i+1/2,j} + u_{i+1/2,j+1}) \)
centered around point \((i + 1/2, j)\), the x-direction momentum equation is solved (J.D. Anderson, JR., Computational Fluid Dynamics, McGraw Hill 1995).

\[
\frac{(\rho u)^{n+1}_{i+1/2,j} - (\rho u)^n_{i+1/2,j}}{\Delta t} = -\left[\frac{(\rho u^2)^n_{i+3/2,j} - (\rho u^2)^n_{i-1/2,j}}{2\Delta x} + \frac{(\rho u\bar{v})^n_{i+1/2,j+1} - (\rho u\bar{v})^n_{i+1/2,j-1}}{2\Delta y}\right] - \frac{p^n_{i+1,j} - p^n_{i,j}}{\Delta x} \\
+ \mu\left[\frac{u^n_{i+3/2,j} - 2u^n_{i+1/2,j} + u^n_{i-1/2,j}}{(\Delta x)^2} + \frac{u^n_{i+1/2,j+1} - 2u^n_{i+1/2,j} + u^n_{i+1/2,j-1}}{(\Delta y)^2}\right]
\]

or

\[
(\rho u)^{n+1}_{i+1/2,j} = (\rho u)^n_{i+1/2,j} + A\Delta t - \frac{\Delta t}{\Delta x}(p^n_{i+1,j} - p^n_{i,j}) \tag{1}
\]

where

\[
A \equiv -\left[\frac{(\rho u^2)^n_{i+3/2,j} - (\rho u^2)^n_{i-1/2,j}}{2\Delta x} + \frac{(\rho u\bar{v})^n_{i+1/2,j+1} - (\rho u\bar{v})^n_{i+1/2,j-1}}{2\Delta y}\right] \\
+ \mu\left[\frac{u^n_{i+3/2,j} - 2u^n_{i+1/2,j} + u^n_{i-1/2,j}}{(\Delta x)^2} + \frac{u^n_{i+1/2,j+1} - 2u^n_{i+1/2,j} + u^n_{i+1/2,j-1}}{(\Delta y)^2}\right]
\]
similarly, centered around point \((i, j + 1/2)\), the \(y\)-direction momentum equation is solved.

\[
\frac{(\rho v)^{n+1}_{i,j+1/2} - (\rho v)^n_{i,j+1/2}}{\Delta t} = -\left[ \frac{(\rho v^2)^n_{i,j+3/2} - (\rho v^2)^n_{i,j-1/2}}{2\Delta y} + \frac{(\rho v\bar{v})^n_{i+1,j+1/2} - (\rho v\bar{v})^n_{i-1,j+1/2}}{2\Delta x} \right] - \frac{p^n_{i,j+1} - p^n_{i,j}}{\Delta y} \\
+ \mu\left[ \frac{v^n_{i,j+3/2} - 2v^n_{i,j+1/2} + v^n_{i,j-1/2}}{(\Delta y)^2} + \frac{v^n_{i+1,j+1/2} - 2v^n_{i,j+1/2} + v^n_{i-1,j+1/2}}{(\Delta x)^2} \right]
\]

or

\[
(\rho v)^{n+1}_{i,j+1/2} = (\rho v)^n_{i,j+1/2} + B\Delta t - \frac{\Delta t}{\Delta y} (p^n_{i,j+1} - p^n_{i,j}) \quad (2)
\]

where

\[
B \equiv \left[ \frac{(\rho v^2)^n_{i,j+3/2} - (\rho v^2)^n_{i,j-1/2}}{2\Delta y} + \frac{(\rho v\bar{v})^n_{i+1,j+1/2} - (\rho v\bar{v})^n_{i-1,j+1/2}}{2\Delta x} \right] \\
+ \mu\left[ \frac{v^n_{i,j+3/2} - 2v^n_{i,j+1/2} + v^n_{i,j-1/2}}{(\Delta y)^2} + \frac{v^n_{i+1,j+1/2} - 2v^n_{i,j+1/2} + v^n_{i-1,j+1/2}}{(\Delta x)^2} \right]
\]
Assume a set of initial guessed \((p^*)^n\), \((\rho u^*)^n\) and \((\rho v^*)^n\) values, equations (1) and (2) can be calculated to yield a new set of \((\rho u^*)^{n+1}\) and \((\rho v^*)^{n+1}\) values,

\[
(\rho u^*)_{i+1/2,j}^{n+1} = (\rho u^*)_{i+1/2,j}^n + A^* \Delta t - \frac{\Delta t}{\Delta x} [(p^*)_{i+1,j}^n - (p^*)_{i,j}^n] \quad (3)
\]

\[
(\rho v^*)_{i,j+1/2}^{n+1} = (\rho v^*)_{i,j+1/2}^n + B^* \Delta t - \frac{\Delta t}{\Delta y} [(p^*)_{i,j+1}^n - (p^*)_{i,j}^n] \quad (4)
\]

let

\[
\begin{align*}
    u' &\equiv u - u^* ; \\
v' &\equiv v - v^* ; \\
p' &\equiv p - p^* \\
A' &\equiv A - A^* ; \\
B' &\equiv B - B^*
\end{align*}
\]

Subtracting eq.(3) from eq.(1) and eq.(4) from eq.(2), it yields

\[
(\rho u')_{i+1/2,j}^{n+1} = (\rho u')_{i+1/2,j}^n + A' \Delta t - \frac{\Delta t}{\Delta x} [(p')_{i+1,j}^n - (p')_{i,j}^n] \quad (5)
\]

\[
(\rho v')_{i,j+1/2}^{n+1} = (\rho v')_{i,j+1/2}^n + B' \Delta t - \frac{\Delta t}{\Delta y} [(p')_{i,j+1}^n - (p')_{i,j}^n] \quad (6)
\]

where \(p', u', v'\) are corrections (修正) of pressure and velocities respectively.
Patankar 建議先將 \( A', B', (pu')^n \) 與 \((pv')^n\) 設為 0，eq. (5) 與 eq. (6)則會分別變為:

\[
(pu')^{n+1}_{i+1/2,j} = -\frac{\Delta t}{\Delta x}[(p')^n_{i+1,j} - (p')^n_{i,j}]
\]

\[
(pv')^{n+1}_{i,j+1/2} = -\frac{\Delta t}{\Delta y}[(p')^n_{i,j+1} - (p')^n_{i,j}]
\]

亦即

\[
(pu)^{n+1}_{i+1/2,j} = (pu^*)^{n+1}_{i+1/2,j} - \frac{\Delta t}{\Delta x}[(p')^n_{i+1,j} - (p')^n_{i,j}]
\]

\[
(pv)^{n+1}_{i,j+1/2} = (pv^*)^{n+1}_{i,j+1/2} - \frac{\Delta t}{\Delta y}[(p')^n_{i,j+1} - (p')^n_{i,j}]
\]

同理，對不同之點上列之關係亦存在，

\[
(pu)^{n+1}_{i-1/2,j} = (pu^*)^{n+1}_{i-1/2,j} - \frac{\Delta t}{\Delta x}[(p')^n_{i,j} - (p')^n_{i-1,j}]
\]

\[
(pv)^{n+1}_{i,j-1/2} = (pv^*)^{n+1}_{i,j-1/2} - \frac{\Delta t}{\Delta y}[(p')^n_{i,j} - (p')^n_{i,j-1}]
\]
點 \((i, j)\) 之質量守恆 (mass conservation)

\[
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0
\]

差分，
\[
\frac{(\rho u)_{i+1/2,j} - (\rho u)_{i-1/2,j}}{\Delta x} + \frac{(\rho v)_{i,j+1/2} - (\rho v)_{i,j-1/2}}{\Delta y} = 0 \tag{13}
\]
將 eqs. (9)-(12) 代入 eq. (13),可得

\[
\begin{align*}
\frac{\{(\rho u^*)_{i+1/2,j}^{n+1} - \frac{\Delta t}{\Delta x} [(p')_{i+1,j}^n - (p')_{i,j}^n]\} - \{(\rho u^*)_{i-1/2,j}^{n+1} - \frac{\Delta t}{\Delta x} [(p')_{i,j}^n - (p')_{i-1,j}^n]\}}{\Delta x} + \\
\frac{\{(\rho v^*)_{i,j+1/2}^{n+1} - \frac{\Delta t}{\Delta y} [(p')_{i,j+1}^n - (p')_{i,j}^n]\} - \{(\rho v^*)_{i,j-1/2}^{n+1} - \frac{\Delta t}{\Delta y} [(p')_{i,j}^n - (p')_{i,j-1}^n]\}}{\Delta y} = 0
\end{align*}
\]

整理後，亦即，

\[
a(p'_{i,j}) + b(p'_{i+1,j}) + b(p'_{i-1,j}) + c(p'_{i,j+1}) + c(p'_{i,j-1}) + d = 0 \tag{14}
\]

其中，

\[
a = 2\left[\frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{(\Delta y)^2}\right] ; \quad b = -\frac{\Delta t}{(\Delta x)^2} ; \quad c = -\frac{\Delta t}{(\Delta y)^2} \\
d = \frac{1}{\Delta x}[(\rho u^*)_{i+1/2,j}^{n+1} - (\rho u^*)_{i-1/2,j}^{n+1}] + \frac{1}{\Delta y}[(\rho v^*)_{i,j+1/2}^{n+1} - (\rho v^*)_{i,j-1/2}^{n+1}]
\]

\[
= \text{mass source term}
\]
Step by Step Procedure for the SIMPLE Algorithm

1. Guessed values of \((p^*)^n\) at all the "pressure" grid points. Also, arbitrarily set values of \((\rho u^*)^n\) and \((\rho v^*)^n\) at the "velocity" grid points.

2. Using equations (3) and (4), solve for \((\rho u^*)^{n+1}\) and \((\rho v^*)^{n+1}\) at all corresponding internal grid points.

3. Substitute these values of \((\rho u^*)^{n+1}\) and \((\rho v^*)^{n+1}\) into equation (14), and solve for \(p'\) values at all appropriate internal grid points.

4. Calculate new "\(p^{n+1}\)" values at all internal grid points.

\[ p^{n+1} = (p^*)^n + p' \]
5. Designate the $p^{n+1}$, $(\rho u^*)^{n+1}$ and $(\rho v^*)^{n+1}$ values obtained above as the new values of $(p^*)^n$, $(\rho u^*)^n$ and $(\rho v^*)^n$ and solve the momentum equations [eqs. (3) and (4)] again.

6. Repeat steps 2-5 until convergence ('d' values at all grid points approach zero) is achieved.

7. When all the 'normalized' mass source terms (normalized 'd' values at all grid points) less than $10^{-4} \sim 10^{-6}$, the solution is considered to be convergent.

8. The larger the selected $\Delta t$ value, the faster the convergence of the solution. But as the selected $\Delta t$ value exceeds a certain level, the solution of the iteration might be divergent.
Dimensionless Governing Equations

Dimensionless groups (無因次化參數):

\[ u^* \equiv u/u_\infty \quad ; \quad v^* \equiv v/u_\infty \quad ; \quad x^* \equiv x/L_y \quad ; \]
\[ y^* \equiv y/L_y \quad ; \quad t^* \equiv t/(L_y / u_\infty) \quad ; \quad P_e^* \equiv P_e/(\rho u_\infty^2) \quad ; \]
\[ \text{Re} \equiv \rho u_\infty (D_h)/\mu = \rho u_\infty (2L_y)/\mu \quad ; \quad T^* \equiv \frac{T - T_w}{T_\infty - T_w} \]

Substituting back into the original equations, it yields

(1) Continuity:
\[ \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \]

(2) Momentum:
\[ \text{x-direction:} \quad \frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = - \frac{\partial P_e^*}{\partial x^*} + \frac{2}{\text{Re}} \left( \frac{\partial^2 u^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^*} \right) \]
\[ \text{y-direction:} \quad \frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = - \frac{\partial P_e^*}{\partial y^*} + \frac{2}{\text{Re}} \left( \frac{\partial^2 v^*}{\partial x^*} + \frac{\partial^2 v^*}{\partial y^*} \right) \]
Example 1: Flow in a Two-Dimensional Channel

the geometry of the scaled problem (2-D channel) is

\[ u^* = 1 \]
\[ v^* = 0 \]

The scaled problem for a 2-D flow is shown in the diagram. The coordinates are scaled as follows:

- \( u^* = 1 \)
- \( v^* = 0 \)
- \( L_x / L_y \)

The flow is directed along the \( x^* \) and \( y^* \) axes, with the total length \( L_x / L_y \) shown in the diagram.
Energy equation:

\[
\frac{\partial T}{\partial t} + \left[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)
\]

where \( \alpha \equiv \frac{k}{\rho c_p} \)
Define: \[ T^* \equiv \frac{T - T_w}{T_\infty - T_w}, \quad x^* = \frac{x}{S}, \quad y^* = \frac{y}{S}, \quad L_x = L, \quad L_y = S, \quad t^* = \frac{t}{(S/U_\infty)} \]

\[ u^* = \frac{u}{U_\infty}, \quad v^* = \frac{v}{U_\infty}, \quad Re = \frac{\rho U_\infty (2S)}{\mu}, \quad Pr = \frac{(\mu / \rho)}{\alpha}, \]

The dimensionless and normalized energy equation can be expressed as follows:

\[ \frac{\partial T^*}{\partial t^*} = - (u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*}) + \frac{2}{Re \ Pr} \left( \frac{\partial^2 T^*}{\partial x^*^2} + \frac{\partial^2 T^*}{\partial y^*^2} \right) \]

\[ = - \left[ \frac{\partial (u^* T^*)}{\partial x^*} + \frac{\partial (v^* T^*)}{\partial y^*} \right] + \frac{2}{Re \ Pr} \left( \frac{\partial^2 T^*}{\partial x^*^2} + \frac{\partial^2 T^*}{\partial y^*^2} \right) \]

\[ T^* = 1 \]

\[ \text{y*} \]

\[ T^* = 0 \]

\[ \text{x*} \]

\[ \text{L}_x / \text{L}_y \]
**Numerical Method**
*(temperature and pressure are located at the same grid point)*

\[
\frac{\partial T^*}{\partial t^*} = \left( \frac{1}{\Delta t^*} \right) [T_{n}^*(i, j) - T_{n-1}^*(i, j)]
\]

\[
\frac{\partial (u^*T^*)}{\partial x^*} = \left( \frac{1}{2\Delta x^*} \right) \left\{ \frac{[u^*(i, j) + u^*(i+1, j)] T^*(i+1, j)}{2} - \frac{[u^*(i-1, j) + u^*(i-2, j)] T^*(i-1, j)}{2} \right\}
\]

\[
\frac{\partial (v^*T^*)}{\partial y^*} = \left( \frac{1}{2\Delta y^*} \right) \left\{ \frac{[v^*(i, j) + v^*(i, j+1)] T^*(i, j+1)}{2} - \frac{[v^*(i, j-1) + v^*(i, j-2)] T^*(i, j-1)}{2} \right\}
\]

\[
\frac{\partial^2 T^*}{\partial x^*^2} = \frac{T^*(i+1, j) - 2T^*(i, j) + T^*(i-1, j)}{(\Delta x^*)^2}
\]

\[
\frac{\partial^2 T^*}{\partial y^*^2} = \frac{T^*(i, j+1) - 2T^*(i, j) + T^*(i, j-1)}{(\Delta y^*)^2}
\]
Substituting back,

\[
T^{*, n+1}_{i, j} = T^{*, n}_{i, j} - \left( \frac{\Delta t^*}{2\Delta x^*} \right) \left\{ \frac{[u^*_{i, j} + u^*_{i+1, j}]}{2} T^{*, n}_{i+1, j} - \frac{[u^*_{i-1, j} + u^*_{i-2, j}]}{2} T^{*, n}_{i-1, j} \right\} \\
- \left( \frac{\Delta t^*}{\Delta y^*} \right) \left\{ \frac{[v^*_{i, j} + v^*_{i, j+1}]}{2} T^{*, n}_{i, j+1} - \frac{[v^*_{i, j-1} + v^*_{i, j-2}]}{2} T^{*, n}_{i, j-1} \right\} \\
+ \left( \frac{2\Delta t^*}{\text{RePr}} \right) \left\{ \frac{T^{*, n}_{i+1, j} - 2T^{*, n}_{i, j} + T^{*, n}_{i-1, j}}{(\Delta x^*)^2} \right\} \\
+ \left( \frac{2\Delta t^*}{\text{RePr}} \right) \left\{ \frac{T^{*, n}_{i, j+1} - 2T^{*, n}_{i, j} + T^{*, n}_{i, j-1}}{(\Delta y^*)^2} \right\}
\]
Equal-size finite-difference grids
Different-size finite-difference grids in main layer and sub-layer
Example 2: Flow in Front of a Block

\[ x^* = \frac{L}{S} \]

\[ y^* = \frac{N}{M} \]
Step by Step Procedure for the SIMPLE Algorithm

1. Starting from \( n = 1 \), guess \( (T^*)^n \) values at all the "pressure" grid points.

2. Using equation (15), solve for \( (T^*)^{n+1} \) at all corresponding internal grid points.

3. Calculate the "d" values at all internal grid points.
   \[
   [d = (T^*)^{n+1} - (T^*)^n]
   \]

4. Designate the \( (T^*)^{n+1} \) values obtained above as the new \( (T^*)^n \) values and solve the energy equation [eq. (15)] again.

5. Repeat steps 2-4 until convergence ('d' values at all grid points approach zero) is achieved.

6. When all the 'd' values at all grid points) less than \(10^{-5}\), the solution is considered to be convergent.