Time-Dependent Conduction:
The Lumped Capacitance Method

Chapter Five
Sections 5.1 thru 5.3
Transient Conduction

• A heat transfer process for which the temperature varies with time, as well as location within a solid.

• It is initiated whenever a system experiences a change in operating conditions and proceeds until a new steady state (thermal equilibrium) is achieved.

• It can be induced by changes in:
  – surface convection conditions ($h, T_\infty$),
  – surface radiation conditions ($h_r, T_{sur}$),
  – a surface temperature or heat flux, and/or
  – internal energy generation.

• Solution Techniques
  – The Lumped Capacitance Method
  – Exact Solutions
  – The Finite-Difference Method
5.1 The Lumped Capacitance Method

- Based on the assumption of a spatially uniform temperature distribution throughout the transient process. Hence, \( T(r, t) \approx T(t) \).

- Why is the assumption never fully realized in practice?

- General Lumped Capacitance Analysis:

  ➢ Consider a general case, which includes convection, radiation and/or an applied heat flux at specified surfaces \( (A_{s,c}, A_{s,r}, A_{s,h}) \), as well as internal energy generation.
First Law: 
\[
\frac{dE_{st}}{dt} = \rho \forall c \frac{dT}{dt} = \dot{E}_\text{in} - \dot{E}_\text{out} + \dot{E}_g
\]

- **Assuming** energy outflow due to convection and radiation and with inflow due to an applied heat flux \( q_s'' \),

\[
\rho \forall c \frac{dT}{dt} = q_s'' A_{s,h} - h A_{s,c} (T - T_\infty) - h_r A_{s,r} (T - T_{sur}) + \dot{E}_g
\]

- Is this expression applicable in situations for which convection and/or radiation provide for energy inflow?

- May \( h \) and \( h_r \) be assumed to be constant throughout the transient process?
- **Special Cases** (Exact Solutions, \( T(0) \equiv T_i \))

  - **Negligible Radiation**

    \[
    (\theta \equiv T - T_\infty, \quad \theta' \equiv \theta - b/a):
    \]

    \[
    a \equiv hA_{s,c} / \rho \forall c \quad b \equiv \left( q''_s A_{s,h} + \dot{E}_g \right) / \rho \forall c
    \]

    The non-homogeneous differential equation is transformed into a homogeneous equation of the form:

    \[
    \frac{d\theta'}{dt} = -a\theta'
    \]
\[
\frac{d\theta'}{\theta'} = -a \ dt
\]

Integrating from \( t=0 \) to any \( t \) and rearranging,

\[
\ln(\theta') = -at + C_1 \quad \Rightarrow \quad \theta' = \exp(-at + C_1)
\]

\[
\theta' = C_2 \exp(-at) \quad \text{or} \quad \theta - b/a = C_2 \exp(-at)
\]

初始條件 \( \square \) at \( t=0 \), \( T = T_i \) \quad \Rightarrow \quad \( C_2 = (T_i - T_\infty) - b/a \)

Hence,

\[
(T - T_\infty) - b/a = [(T_i - T_\infty) - b/a] \exp(-at)
\]

\[
\frac{T - T_\infty}{T_i - T_\infty} = \frac{b/a}{T_i - T_\infty} + [1 - \frac{b/a}{T_i - T_\infty}] \exp(-at)
\]

or,

\[
\frac{T - T_\infty}{T_i - T_\infty} = \exp(-at) + \frac{b/a}{T_i - T_\infty} \left[ 1 - \exp(-at) \right] \quad (5.25)
\]

To what does the foregoing equation reduce as steady state is approached?

How else may the steady-state solution be obtained?
Negligible Radiation and Source Terms \( (h \gg h_r, \dot{E}_g = 0, q_s'' = 0) \): 

\[
\rho \nabla c \frac{dT}{dt} = -hA_{s,c} (T - T_\infty) \tag{5.2}
\]

\[
\frac{\rho \nabla c}{hA_{s,c}} \int_{\theta_i}^{\theta} \frac{d \theta}{\theta} = -\int_0^t dt
\]

\[
\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp \left[ -\left( \frac{hA_{s,c}}{\rho \nabla c} \right) t \right] = \exp \left[ -\frac{t}{\tau_i} \right]
\]

**Figure 5.1** Cooling of a hot metal forging.
The thermal time constant is defined as

\[ \tau_t \equiv \frac{1}{hA_{s,c}} \left( \rho \forall c \right) \tag{5.7} \]

The change in thermal energy storage due to the transient process is

\[ \Delta E_{st} \equiv -Q = - \int_0^t E_{out} dt = -hA_{s,c} \int_0^t \theta dt = -(\rho \forall c) \theta_i \left[ 1 - \exp \left( -\frac{t}{\tau_t} \right) \right] \tag{5.8} \]
Negligible Convection and Source Terms \( h_r >> h, \quad \dot{E}_g = 0, \quad q''_s = 0 \):

Assuming radiation exchange with large surroundings,

\[
\rho \nabla c \frac{dT}{dt} = -\varepsilon A_{s,r} \sigma \left( T^4 - T_{\text{sur}}^4 \right)
\]

\[
\frac{\varepsilon A_{s,r} \sigma}{\rho \nabla c} \int_0^t dt = \int_{T_i}^T \frac{dT}{T_{\text{sur}}^4 - T^4}
\]

\[
t = \frac{\rho \nabla c}{4 \varepsilon A_{s,r} \sigma T_{\text{sur}}^3} \left\{ \ln \left| \frac{T_{\text{sur}} + T}{T_{\text{sur}} - T} \right| - \ln \left| \frac{T_{\text{sur}} + T_i}{T_{\text{sur}} - T_i} \right| \right. \\
+ 2 \left[ \tan^{-1} \left( \frac{T}{T_{\text{sur}}} \right) - \tan^{-1} \left( \frac{T_i}{T_{\text{sur}}} \right) \right] \right\}
\]

(5.18)

Result necessitates implicit evaluation of \( T(t) \).
5.2 Validity of The Lumped Capacitance Method

- The Biot Number: The first of many dimensionless parameters to be considered.

  ➢ Definition:

  \[ \text{Bi} \equiv \text{Biot number} = \frac{hL_c}{k} \]

  \( h \rightarrow \) convection or radiation coefficient

  \( k \rightarrow \) thermal conductivity of the solid

  \( L_c \rightarrow \) characteristic length of the solid (\( \forall / A_s \) or coordinate associated with maximum spatial temperature difference)

  ➢ Physical Interpretation:

  \[ Bi = \frac{L_c}{kA_s} \sim \frac{R_{\text{cond}}}{R_{\text{conv}}} \sim \frac{\Delta T_{\text{solid}}}{\Delta T_{\text{solid / fluid}}} \]

  ➢ Criterion for Applicability of Lumped Capacitance Method: \( Bi << 1 \)
Physical Interpretation:

\[ \frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_{\infty}} = \frac{L}{kA_s} = \frac{R_{\text{cond}}}{R_{\text{conv}}} = \frac{hL}{k} = Bi \]

- **Bi** \(\approx 0.1\) (use Lumped Heat Capacitance method conditions)

\[ Bi = \frac{hL_c}{k} \]

\[ L_c = \frac{V}{A_s} \]

- **Plate**: \( L_c = L \)
- **Cylinder**: \( L_c = r_o / 2 \)
- **Sphere**: \( L_c = r_o / 3 \) (consider \( L_c = r_o \))
Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

\[
\frac{T - T_\infty}{T_i - T_\infty} = \exp \left[ -\left(\frac{hA_{s,c}}{\rho c L_c}\right)t \right] = \exp \left[ -\left(\frac{hL_c}{k}\frac{A_{s,c} k}{\rho c L_c}t\right) \right] = \exp \left[ -\left(\frac{hL_c}{k}\frac{k}{\rho c L_c}t\right) \right] = \exp \left[ -\left(\frac{hL_c}{k}\frac{\alpha t}{L_c^2}\right) \right] \equiv \exp \left[ -(Bi)(Fo) \right]
\]

\[
Fo = \frac{\alpha t}{L_c^2} = \text{Fourier number}
\]
A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is $h = 400 \text{ W/m}^2 \cdot \text{K}$, and the junction thermophysical properties are $k = 20 \text{ W/m} \cdot \text{K}$, $c = 400 \text{ J/kg} \cdot \text{K}$, and $\rho = 8500 \text{ kg/m}^3$. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?
Analysis:

1. Because the junction diameter is unknown, it is not possible to begin the solution by determining whether the criterion for using the lumped capacitance method, Equation 5.10, is satisfied. However, a reasonable approach is to use the method to find the diameter and to then determine whether the criterion is satisfied. From Equation 5.7 and the fact that \( A_s = \pi D^2 \) and \( V = \pi D^3/6 \) for a sphere, it follows that

\[
\tau = \frac{1}{h\pi D^2} \times \frac{\rho \pi D^3}{6}c
\]

Rearranging and substituting numerical values,

\[
D = \frac{6h\tau}{\rho c} = \frac{6 \times 400 \text{ W/m}^2 \cdot \text{K} \times 1 \text{ s}}{8500 \text{ kg/m}^3 \times 400 \text{ J/kg} \cdot \text{K}} = 7.06 \times 10^{-4} \text{ m}
\]

With \( L_c = r_o/3 \) it then follows from Equation 5.10 that

\[
Bi = \frac{h(r_o/3)}{k} = \frac{400 \text{ W/m}^2 \cdot \text{K} \times 3.53 \times 10^{-4} \text{ m}}{3 \times 20 \text{ W/m} \cdot \text{K}} = 2.35 \times 10^{-3}
\]

Accordingly, Equation 5.10 is satisfied (for \( L_c = r_o \), as well as for \( L_c = r_o/3 \)) and the lumped capacitance method may be used to an excellent approximation.
2. From Equation 5.5 the time required for the junction to reach $T = 199^\circ C$ is

\[
t = \frac{\rho(\pi D^3/6)c}{h(\pi D^2)} \ln \frac{T_i - T_\infty}{T - T_\infty} = \frac{\rho D c}{6 h} \ln \frac{T_i - T_\infty}{T - T_\infty}
\]

\[
t = \frac{8500 \text{ kg/m}^3 \times 7.06 \times 10^{-4} \text{ m} \times 400 \text{ J/kg} \cdot \text{K}}{6 \times 400 \text{ W/m}^2 \cdot \text{K}} \ln \frac{25 - 200}{199 - 200}
\]

\[
t = 5.2 \text{ s} \approx 5\tau_i
\]

Comments: Heat transfer due to radiation exchange between the junction and the surroundings and conduction through the leads would affect the time response of the junction and would, in fact, yield an equilibrium temperature that differs from $T_\infty$. 
EXAMPLE 5.2

Consider the thermocouple and convection conditions of Example 5.1, but now allow for radiation exchange with the walls of a duct that encloses the gas stream. If the duct walls are at 400°C and the emissivity of the thermocouple bead is 0.9, calculate the steady-state temperature of the junction. Also, determine the time for the junction temperature to increase from an initial condition of 25°C to a temperature that is within 1°C of its steady-state value.

Gas stream

- $T_\infty = 200^\circ C$
- $h = 400 \text{ W/m}^2\cdot\text{K}$

Hot duct wall,

$T_{\text{sur}} = 400^\circ C$

Junction, $T(t)$

- $T_i = 25^\circ C$, $D = 0.7 \text{ mm}$
- $\rho = 8500 \text{ kg/m}^3$
- $c = 400 \text{ J/kg}\cdot\text{K}$
- $\varepsilon = 0.9$
Analysis:

1. For steady-state conditions, the energy balance on the thermocouple junction has the form

\[ \dot{E}_{in} - \dot{E}_{out} = 0 \]

Recognizing that net radiation to the junction must be balanced by convection from the junction to the gas, the energy balance may be expressed as

\[ [\varepsilon \sigma (T^4_{\text{sur}} - T^4) - h(T - T_\infty)]A_s = 0 \]

Substituting numerical values, we obtain

\[ T = 218.7^\circ C \]

2. The temperature-time history, \( T(t) \), for the junction, initially at \( T(0) = T_i = 25^\circ C \), follows from the energy balance for transient conditions,

\[ \dot{E}_{in} - \dot{E}_{out} = \dot{E}_{st} \]

From Equation 5.15, the energy balance may be expressed as

\[ -[h(T - T_\infty) + \varepsilon \sigma (T^4 - T^4_{\text{sur}})]A_s = \rho Vc \frac{dT}{dt} \]

The solution to this first-order differential equation can be obtained by numerical integration, giving the result, \( T(4.9 \text{ s}) = 217.7^\circ C \). Hence, the time required to reach a temperature that is within 1\(^\circ\)C of the steady-state value is

\[ t = 4.9 \text{ s.} \]
Example 5.3

A 3-mm-thick panel of aluminum alloy \((k = 177 \text{ W/m} \cdot \text{K}, \ c = 875 \text{ J/kg} \cdot \text{K}, \ \text{and} \ \rho = 2770 \text{ kg/m}^3)\) is finished on both sides with an epoxy coating that must be cured at or above \(T_c = 150^\circ\text{C}\) for at least 5 min. The production line for the curing operation involves two steps: (1) heating in a large oven with air at \(T_{\infty,o} = 175^\circ\text{C}\) and a convection coefficient of \(h_o = 40 \text{ W/m}^2 \cdot \text{K}\), and (2) cooling in a large chamber with air at \(T_{\infty,c} = 25^\circ\text{C}\) and a convection coefficient of \(h_c = 10 \text{ W/m}^2 \cdot \text{K}\). The heating portion of the process is conducted over a time interval \(t_s\), which exceeds the time \(t_c\) required to reach \(150^\circ\text{C}\) by 5 min \((t_s = t_c + 300 \text{ s})\). The coating has an emissivity of \(\varepsilon = 0.8\), and the temperatures of the oven and chamber walls are \(175^\circ\text{C}\) and \(25^\circ\text{C}\), respectively. If the panel is placed in the oven at an initial temperature of \(25^\circ\text{C}\) and

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Diagram:

- 2L = 3 mm
- \(T_{\text{sur, o}} = 175^\circ\text{C}\)
- \(T_{\infty,o} = 175^\circ\text{C}\)
- \(h_o, T_{\infty,o} = 175^\circ\text{C}\)
- Epoxy, \(\varepsilon = 0.8\)
- Aluminum, \(T(0) = T_{i,o} = 25^\circ\text{C}\)
- Step 1: Heating \((0 \leq t \leq t_c)\)

- \(T_{\text{sur, c}} = 25^\circ\text{C}\)
- \(h_c, T_{\infty,c} = 25^\circ\text{C}\)
- \(T(t_f) = 37^\circ\text{C}\)
- Step 2: Cooling \((t_c < t \leq t_f)\)
**Analysis:** To assess the validity of the lumped capacitance approximation, we begin by calculating Biot numbers for the heating and cooling processes.

\[ Bi_h = \frac{h_c L}{k} = \frac{(40 \text{ W/m}^2 \cdot \text{K})(0.0015 \text{ m})}{177 \text{ W/m} \cdot \text{K}} = 3.4 \times 10^{-4} \]

\[ Bi_c = \frac{h_c L}{k} = \frac{(10 \text{ W/m}^2 \cdot \text{K})(0.0015 \text{ m})}{177 \text{ W/m} \cdot \text{K}} = 8.5 \times 10^{-5} \]

Hence the lumped capacitance approximation is excellent.

To determine whether radiation exchange between the panel and its surroundings should be considered, the radiation heat transfer coefficient is determined from Equation 1.9. A representative value of \( h_r \) for the heating process is associated with the cure condition, in which case

\[ h_{r,o} = \varepsilon \sigma (T_c + T_{\text{sur,o}})(T_c^2 + T_{\text{sur,o}}^2) \]

\[ = 0.8 \times 5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4(423 + 448)K(423^2 + 448^2)K^2 \]

\[ = 15 \text{ W/m}^2 \cdot \text{K} \]
Using $T_c = 150^\circ C$ with $T_{\text{sur,c}} = 5^\circ C$ for the cooling process, we also obtain $h_{r,c} = 8.8 \text{ W/m}^2 \cdot \text{K}$. Since the values of $h_{r,o}$ and $h_{r,c}$ are comparable to those of $h_o$ and $h_c$, respectively, radiation effects must be considered.

With $V = 2LA_s$ and $A_{s,c} = A_{s,r} = 2A_s$, Equation 5.15 may be expressed as

$$
\int_{T_i}^{T}dT = T(t) - T_i = -\frac{1}{\rho c L} \int_{0}^{t} [h(T - T_\infty) + \epsilon \sigma (T^4 - T_{\text{sur}}^4)]dt
$$

Selecting a suitable time increment $\Delta t$, the right-hand side of this equation may be evaluated numerically to obtain the panel temperature at $t = \Delta t, 2\Delta t, 3\Delta t$, and so on. At each new step of the calculation, the value of $T$ computed from the previous time step is used in the integrand. Selecting $\Delta t = 10 \text{ s}$, calculations for the heating process are extended to $t_e = t_c + 300 \text{ s}$, which is 5 min beyond the time required for the panel to reach $T_c = 150^\circ C$. At $t_e$ the cooling process is initiated and continued until the panel temperature reaches $37^\circ C$ at $t = t_r$. The integration was performed using a fourth-order Runge-Kutta scheme, and results of the calculations are plotted as follows:
The total time for the two-step process is

\[ t_t = 989 \text{ s} \]

with intermediate times of \( t_c = 124 \text{ s} \) and \( t_e = 424 \text{ s} \).
Problem 5.12: Charging a thermal energy storage system consisting of a packed bed of aluminum spheres.

**KNOWN:** Diameter, density, specific heat and thermal conductivity of aluminum spheres used in packed bed thermal energy storage system. Convection coefficient and inlet gas temperature.

**FIND:** Time required for sphere at inlet to acquire 90% of maximum possible thermal energy and the corresponding center temperature.

**Schematic:**

Aluminum sphere
- $D = 75$ mm, $T_i = 25^\circ$C
- $\rho = 2700$ kg/m$^3$
- $c = 950$ J/kg-K
- $k = 240$ W/m-K

Inlet gas
- $T_{g,i} = 300^\circ$C
- $h = 75$ W/m$^2$-K
**ASSUMPTIONS:** (1) Negligible heat transfer to or from a sphere by radiation or conduction due to contact with other spheres, (2) Constant properties.

**ANALYSIS:** To determine whether a lumped capacitance analysis can be used, first compute $Bi = \frac{h(r_o/3)}{k} = 75 \text{ W/m}^2 \cdot \text{K} \ (0.025 \text{ m})/150 \text{ W/m} \cdot \text{K} = 0.013 << 1$.

Hence, the lumped capacitance approximation may be made, and a uniform temperature may be assumed to exist in the sphere at any time.

From Eq. 5.8a, achievement of 90% of the maximum possible thermal energy storage corresponds to

$$\frac{\Delta E_{st}}{\rho cV \theta_i} = 0.90 = 1 - \exp\left(-\frac{t}{\tau_t}\right)$$

$$\tau_t = \frac{\rho Vc}{hA_s} = \frac{\rho Dc}{6h} = \frac{2700 \text{ kg/m}^3 \times 0.075 \text{ m} \times 950 \text{ J/kg} \cdot \text{K}}{6 \times 75 \text{ W/m}^2 \cdot \text{K}} = 427 \text{ s}.$$

$$t = -\tau_t \ln(0.1) = 427 \text{ s} \times 2.30 = 984 \text{ s}$$
From Eq. (5.6), the corresponding temperature at any location in the sphere is

\[
T(984s) = 300°C - 275°C \exp\left(-6 \times 75 \text{ W/m}^2 \cdot \text{K} \times 984s / 2700 \text{ kg/m}^3 \times 0.075m \times 950 \text{ J/kg} \cdot \text{K}\right) \\
T(984s) = T_{g,i} + \left(T_i - T_{g,i}\right) \exp\left(-6ht / \rho Dc\right)
\]

\[
T(984s) = 272.5°C
\]

If the product of the density and specific heat of copper is \((\rho c)_{\text{Cu}} \approx 8900 \text{ kg/m}^3 \times 400 \text{ J/kg} \cdot \text{K} = 3.56 \times 10^6 \text{ J/m}^3 \cdot \text{K}\), is there any advantage to using copper spheres of equivalent diameter in lieu of aluminum spheres?

Does the time required for a sphere to reach a prescribed state of thermal energy storage change with increasing distance from the bed inlet? If so, how and why?