Steady-State, One-Dimensional Conduction

Chapter Three
Section 3.5, Appendix C
3.5 Conduction with Thermal Energy Generation

- Involves a local (volumetric) source of thermal energy due to conversion from another form of energy in a conducting medium.

- The source may be uniformly distributed, as in the conversion from electrical to thermal energy (Ohmic heating):

\[
q = \frac{\dot{E}_g}{\forall} = \frac{I^2R_c}{\forall} \tag{3.38}
\]

or it may be non-uniformly distributed, as in the absorption of radiation passing through a semi-transparent medium. For a plane wall,

\[
q \propto \exp(-\alpha x)
\]

- Generation affects the temperature distribution in the medium and causes the heat rate to vary with location, thereby precluding (排除) inclusion of the medium in a thermal circuit.
The Plane Wall

- Consider one-dimensional, steady-state conduction in a plane wall of constant $k$, uniform generation, and asymmetric surface conditions:

- **Heat Equation:**

\[
\frac{d}{dx} \left( k \frac{dT}{dx} \right) + q = 0 \Rightarrow \frac{d^2T}{dx^2} + \frac{q}{k} = 0 \quad (3.39)
\]

  Is the heat flux $q''$ independent of $x$?

- **General Solution:**

\[
T(x) = -\left( \frac{\dot{q}}{2k} \right) x^2 + C_1 x + C_2
\]

What is the form of the temperature distribution for $\dot{q} = 0$? $\dot{q} > 0$? $\dot{q} < 0$?

How does the temperature distribution change with increasing $\dot{q}$?
Symmetric Surface Conditions or One Surface Insulated:

- What is the temperature gradient at the centerline or the insulated surface?
- Why does the magnitude of the temperature gradient increase with increasing x?
- Temperature Distribution:
  \[ T(x) = \frac{qL^2}{2k} \left( 1 - \frac{x^2}{L^2} \right) + T_s \]  
  \( (3.42) \)
- How do we determine \( T_s \)?
  Overall energy balance on the wall
  \[
  -\dot{E}_{\text{out}} + \dot{E}_g = 0
  \]
  \[
  -hA_s (T_s - T_\infty) + q A_s L = 0
  \]
  \[
  T_s = T_\infty + \frac{qL}{h} \]  
  \( (3.46) \)
- How do we determine the heat rate at \( x = L \)?
Radial Systems

Cylindrical (Tube) Wall

Solid Cylinder (Circular Rod)

Spherical Wall (Shell)

Solid Sphere

- Heat Equations:
  - Cylindrical
    \[ \frac{1}{r} \frac{d}{dr} \left( kr \frac{dT}{dr} \right) + \dot{q} = 0 \]
  - Spherical
    \[ \frac{1}{r^2} \frac{d}{dr} \left( kr^2 \frac{dT}{dr} \right) + \dot{q} = 0 \]
• Solution for **Uniform Generation in a Solid Sphere of Constant \( k \) with Convection Cooling:**

**Temperature Distribution**

\[
kr^2 \frac{dT}{dr} = -\frac{q r^3}{3} + C_1
\]

\[
T = -\frac{q r^2}{6k} - \frac{C_1}{r} + C_2
\]

\[
\frac{dT}{dr} \bigg|_{r=0} = 0 \quad \rightarrow \quad C_1 = 0
\]

\[
T(r_o) = T_s \quad \rightarrow \quad C_2 = T_s + \frac{q r_o^2}{6k}
\]

\[
T(r) = \frac{q r_o^2}{6k} \left( 1 - \frac{r^2}{r_o^2} \right) + T_s
\]

**Surface Temperature**

**Overall energy balance:**

\[
-\dot{E}_{out} + \dot{E}_g = 0 
\]

\[
\rightarrow \quad T_s = T_\infty + \frac{q r_o}{3h}
\]

Or from a surface energy balance:

\[
\dot{E}_{in} - \dot{E}_{out} = 0 
\]

\[
\rightarrow \quad q_{\text{cond}} (r_o) = q_{\text{conv}} 
\]

\[
\rightarrow \quad T_s = T_\infty + \frac{q r_o}{3h}
\]

• A summary of temperature distributions is provided in Appendix C for plane, cylindrical and spherical walls, as well as for solid cylinders and spheres. Note how boundary conditions are specified and how they are used to obtain surface temperatures.
**Example 3.7**

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6 \text{ W/m}^3$, $k_A = 75 \text{ W/m} \cdot \text{K}$, and thickness $L_A = 50 \text{ mm}$. The wall material B has no generation with $k_B = 150 \text{ W/m} \cdot \text{K}$ and thickness $L_B = 20 \text{ mm}$. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30^\circ\text{C}$ and $h = 1000 \text{ W/m}^2 \cdot \text{K}$.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.

2. Determine the temperature $T_0$ of the insulated surface and the temperature $T_2$ of the cooled surface.
Analysis:
1. From the prescribed physical conditions, the temperature distribution in the composite is known to have the following features, as shown:
   (a) Parabolic in material A.
   (b) Zero slope at insulated boundary.
   (c) Linear in material B.
   (d) Slope change $= k_B/k_A = 2$ at interface.
   The temperature distribution in the water is characterized by
   (e) Large gradients near the surface.
2. The outer surface temperature $T_2$ may be obtained by performing an energy balance on a control volume about material B. Since there is no generation in this material, it follows that, for steady-state conditions and a unit surface area, the heat flux into the material at $x = L_A$ must equal the heat flux from the material due to convection at $x = L_A + L_B$. Hence

$$q'' = h(T_2 - T_\infty)$$  \hspace{1cm} (1)

The heat flux $q''$ may be determined by performing a second energy balance on a control volume about material A. In particular, since the surface at $x = 0$ is adiabatic, there is no inflow and the rate at which energy is generated must equal the outflow. Accordingly, for a unit surface area,

$$\dot{q}L_A = q''$$  \hspace{1cm} (2)

Combining Equations 1 and 2, the outer surface temperature is

$$T_2 = T_\infty + \frac{\dot{q}L_A}{h}$$

$$T_2 = 30^\circC + \frac{1.5 \times 10^6 \text{ W/m}^3 \times 0.05 \text{ m}}{1000 \text{ W/m}^2 \cdot \text{K}} = 105^\circC$$

From Equation 3.43 the temperature at the insulated surface is

$$T_0 = \frac{\dot{q}L_A^2}{2k_A} + T_1$$  \hspace{1cm} (3)

where $T_1$ may be obtained from the following thermal circuit:

```
q'' ----> T1 ----> T2 ----> T\infty
R_{cond,A} \hspace{1cm} R_{conv}
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That is,

\[ T_1 = T_\infty + (R''_{\text{cond,B}} + R''_{\text{conv}}) q'' \]

where the resistances for a unit surface area are

\[ R''_{\text{cond,B}} = \frac{L_B}{k_B} \quad R''_{\text{conv}} = \frac{1}{h} \]

Hence,

\[ T_1 = 30^\circ C + \left( \frac{0.02 \text{ m}}{150 \text{ W/m} \cdot \text{K}} + \frac{1}{1000 \text{ W/m}^2 \cdot \text{K}} \right) \]

\[ \times 1.5 \times 10^6 \text{ W/m}^3 \times 0.05 \text{ m} \]

\[ T_1 = 30^\circ C + 85^\circ C = 115^\circ C \]

Substituting into Equation 3,

\[ T_0 = \frac{1.5 \times 10^6 \text{ W/m}^3 (0.05 \text{ m})^2}{2 \times 75 \text{ W/m} \cdot \text{K}} + 115^\circ C \]

\[ T_0 = 25^\circ C + 115^\circ C = 140^\circ C \]
\( h = 200 \text{ W/m}^2\cdot\text{K} \)
Consider a long solid tube, insulated at the outer radius $r_2$ and cooled at the inner radius $r_1$, with uniform heat generation $\dot{q}$ (W/m$^3$) within the solid.

1. Obtain the general solution for the temperature distribution in the tube.

2. In a practical application a limit would be placed on the maximum temperature that is permissible at the insulated surface ($r = r_2$). Specifying this limit as $T_{x,2}$, identify appropriate boundary conditions that could be used to determine the arbitrary constants appearing in the general solution. Determine these constants and the corresponding form of the temperature distribution.

3. Determine the heat removal rate per unit length of tube.

4. If the coolant is available at a temperature $T_\infty$, obtain an expression for the convection coefficient that would have to be maintained at the inner surface to allow for operation at prescribed values of $T_{x,2}$ and $\dot{q}$.
Analysis:

1. To determine $T(r)$, the appropriate form of the heat equation, Equation 2.24, must be solved. For the prescribed conditions, this expression reduces to Equation 3.49, and the general solution is given by Equation 3.51. Hence, this solution applies in a cylindrical shell, as well as in a solid cylinder (Figure 3.10).

2. Two boundary conditions are needed to evaluate $C_1$ and $C_2$, and in this problem it is appropriate to specify both conditions at $r_2$. Invoking the prescribed temperature limit,

$$T(r_2) = T_{s,2}$$

and applying Fourier's law, Equation 3.24, at the adiabatic outer surface

$$\left. \frac{dT}{dr} \right|_{r_2} = 0$$

Using Equations 3.51 and 1, it follows that

$$T_{s,2} = -\frac{q}{4k} r_2^2 + C_1 \ln r_2 + C_2$$

Similarly, from Equations 3.50 and 2

$$0 = -\frac{q}{2k} r_2^2 + C_1$$
Hence, from Equation 4,

\[ C_1 = \frac{q}{2k} r_2^2 \]  \hspace{1cm} (5)

and from Equation 3

\[ C_2 = T_{s,2} + \frac{q}{4k} r_2^2 - \frac{q}{2k} r_2^2 \ln r_2 \]  \hspace{1cm} (6)

Substituting Equations 5 and 6 into the general solution, Equation 3.51, it follows that

\[ T(r) = T_{s,2} + \frac{q}{4k} (r_2^2 - r^2) - \frac{q}{2k} r_2^2 \ln \frac{r_2}{r} \]  \hspace{1cm} (7)

3. The heat removal rate may be determined by obtaining the conduction rate at \( r_1 \) or by evaluating the total generation rate for the tube. From Fourier's law

\[ q_r' = -k2\pi r \frac{dT}{dr} \]

Hence, substituting from Equation 7 and evaluating the result at \( r_1 \),

\[ q_r'(r_1) = -k2\pi r_1 \left( -\frac{q}{2k} r_1 + \frac{q}{2k} \frac{r_2^2}{r_1} \right) = -\pi q (r_2^2 - r_1^2) \]  \hspace{1cm} (8)

Alternatively, because the tube is insulated at \( r_2 \), the rate at which heat is generated in the tube must equal the rate of removal at \( r_1 \). That is, for a control volume about the tube, the energy conservation requirement, Equation 1.11c, reduces to \( \dot{E}_g - \dot{E}_{out} = 0 \), where \( \dot{E}_g = \dot{q} \pi (r_2^2 - r_1^2) L \) and \( \dot{E}_{out} = q_{\text{cond}}' L = -q_r'(r_1)L \). Hence

\[ q_r'(r_1) = -\pi \dot{q} (r_2^2 - r_1^2) \]  \hspace{1cm} (9)
4. Applying the energy conservation requirement, Equation 1.12, to the inner surface, it follows that

\[ q'_{\text{cond}} = q'_{\text{conv}} \]

or

\[ \pi \dot{q} (r_2^2 - r_1^2) = h 2 \pi r_1 (T_{s,1} - T_\infty) \]

Hence

\[ h = \frac{\dot{q} (r_2^2 - r_1^2)}{2r_1 (T_{s,1} - T_\infty)} \] (10)

where \( T_{s,1} \) may be obtained by evaluating Equation 7 at \( r = r_1 \).